Nouvelle tendance pour le préconditionnement des équations intégrales en électromagnétisme

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OUTLINES

Context and motivations

- The application we are interested on is the computation of the Radar Cross Section in time-harmonic domain.
- Integral equation method
- High frequency regime
- In consequence, we have to face some linear systems which are
  - Full
  - Huge (few thousands or millions of unknowns)
  - Poorly conditioned
- In certains situations (deep cavity for example), algebraic (classical) preconditioners are not efficient

$\Rightarrow$We aim new preconditioning techniques getting their strength from the analysis of the continuous equations from which are derived the linear systems we have to solve.

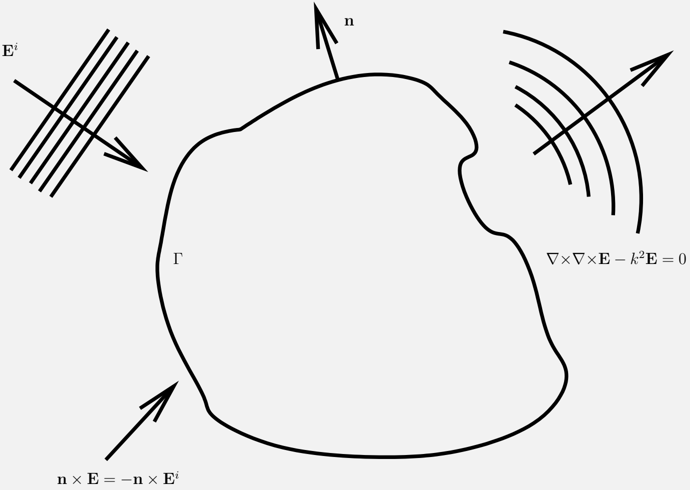
Two axis of research

- Analytical preconditioners for classical equations
  - Get an efficient preconditioner of a given equation from the knowledge of its parametrix
  - First contributions were given by McLean-Tran and Steinback-Wendland (1997)
  - Christiansen in 2001 gave an application of this strategy to build a "Calderon preconditioner" of the EFIE
- "Inherently preconditioned" integral equations
  - Deals with the construction of an indirect integral equation which, after discretization, is natively well-conditioned.
  - Generalization of a little forgotten equation due to Mautz-Harrigton in 1979

Our goal

- We aim an efficient resolution of problems with an impedance or a transmission condition
- An inherently pre-conditioned integral equation : no preconditioner is needed after discretization
- The equation takes inspiration from a previous integral formalism
  - F. Alouges, S. Borel, and D. P. Levadoux, A stable well-conditioned integral equation for electromagnetism scattering, J. Comp. Appl. Math 204 (2007), 440-451.
  - S. Borel, D. P. Levadoux, and F. Alouges, A new well-conditioned integral formulation for Maxwell equations in three-dimensions, IEEE Trans. Antennas Propag. 53 (2005), no. 9, 2995-3004.
  - M. Darbas and X. Antoine, Generalized combined field integral equations for the iterative solution of the Helmholtz equation in three dimensions, M2AN Vol. 41 (2007), 147–167.

The Boundary Value Problem (BVP)



The field/source integral equations

- **FIELD** integral equation (direct method) :

  Consists to write constraints on the Cauchy data of the solution (i.e. the FIELD) of the BVP through an equation written on the boundary of the scatterer. Solution of the BVP is found by application of a "representation theorem" (e.g. Stratton-Chu or Green formula)

- **SOURCE** integral equation (indirect method)

  Parametrization of the space of the radiating waves W^+ in which the solution of the BVP is supposed to live :

$$\mathcal{D}'(\Gamma) \quad \overset{\nu}{\longrightarrow} \quad W^+$$

  The boundary condition required by the BVP leads to an IE whose unknown is an abstract current or a SOURCE :

$$\text{Find } \mathbf{u} \in \mathcal{D}'(\mathbf{\Gamma}) \quad \text{such that} \quad \mathbf{n} \times \nu\mathbf{u} = -\mathbf{n} \times \mathbf{E^i}$$

Classical integral equations

- Applying 1 or 2 curl operators after a convolution with the Green kernel of the Helmholtz equation (vector potential) leads to two fundamental potentials on which all classical integral equations of electromagnetism are founded

$$\mathcal{T} = \frac{1}{ik}\nabla\times\nabla\times G, \quad \mathcal{K} = \nabla\times G \quad \text{with} \quad G\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\int_\Gamma \frac{e^{ik\|\mathbf{x}-\mathbf{y}\|}}{\|\mathbf{x}-\mathbf{y}\|}\mathbf{u}(\mathbf{y})d\mathbf{y}$$

- In the engineering culture, field integral equations are very popular

$$\text{EFIE} \quad \mathcal{T}_{\tan}\mathbf{J} = -\mathbf{E}^i_{\tan}$$

$$\text{MFIE} \quad (\mathrm{Id} + \mathbf{n}\times\mathcal{K})\mathbf{J} = \mathbf{n}\times\mathbf{H}^i$$

$$\text{CFIE} \quad (1-\alpha)\text{EFIE} + \alpha\text{MFIE}, \ \alpha\in[0,1]$$

- The source integral equation proposed by Mautz and Harrington (1979) will be the starting point of the new SIE we want to present here. M.-H. write a combined source integral equation using the parameterization of W^+ by a combination of $\mathcal{T}$ and $\mathcal{K}$: $\alpha\mathcal{T} - \mathcal{K}$

$$\text{CSIE} \quad \mathbf{n}\times(\alpha\mathcal{T} - \mathcal{K})\mathbf{u} = -\mathbf{n}\times\mathbf{E}^i$$

Some properties of classical integral equations

Equation	Well-posed at any frequency	Compact perturbation od identity
EFIE	No	No
MFIE	No	yes
CFIE	yes	No
CSIE	yes	No

Our goal : to build an equation being well-posed at any frequency and being a compact perturbation of identity

PLAN

General writing of a scattering problem

All boundary (or transmission) problems we plan to solve, read formally as

The abstract problem

Find $u \in W$ such that $\gamma\, u = u_0$

- $u_0 \in \mathcal{D}'(\Gamma)$ is a distribution on the boundary Γ of a compact set D
- W is a functional space of admissible wave solutions usually defined on $\mathbb{R}^3 \setminus D$ or $\mathbb{R}^3 \setminus \Gamma$
- γ is the boundary condition trace operator of the scattering problem

PLAN

Calderón potential

- Beside the boundary condition trace operator γ, we have to give us a Cauchy data trace operator γ_c
- It means we suppose to have a reconstruction formula able to rebuild any field $w \in W$ from the knowledge of its Cauchy data $\gamma_c w$
- For instance, Green or Stratton-Chu formulas state that

$$w = \mathcal{C}(\gamma_c w)$$

  where $\mathcal{C}$ is a potential $\mathcal{D}'(\Gamma) \to W$
- We name $\mathcal{C}$ the Calderón potential of the scattering problem

PLAN

The boundary operator R

- We are now able to define the (optimal) regularizing operator R
- The initial BVP is well-posed

$$\text{Find } w \in W \text{ such that } \gamma w = u_0$$

- There exists an operator R linking the boundary condition traces of a field to its Cauchy data

$$R : \gamma w \mapsto \gamma_c w$$

- By definition, R verifies a crucial relation

$$\gamma \mathcal{C} R = \mathrm{Id}$$

PLAN

A new class of boundary integral equations

- Let $\widetilde{R}$ be an approximation of R
- We search the solution w of the BVP under the form

$$w = \mathcal{C}\widetilde{R}u$$

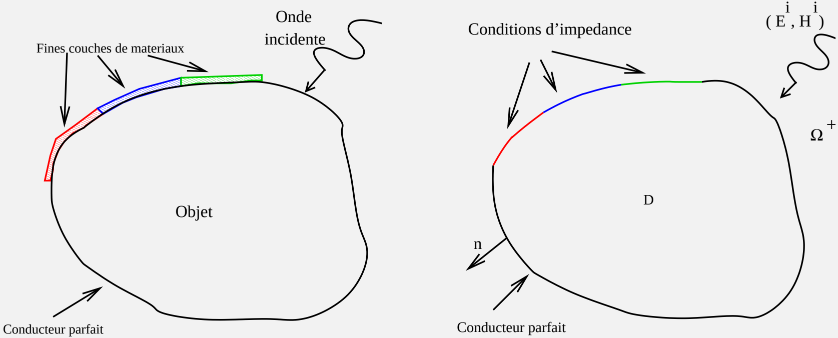
- The resulting source (or indirect) integral equation is

$$\text{Find } u \in \mathcal{D}'(\Gamma) \text{ such that } \gamma\mathcal{C}\widetilde{R}u = u_0 \tag{1}$$

- If $\widetilde{R} = R$ the new equation is trivial (the crucial relation $\gamma\mathcal{C}R = \mathrm{Id}$)
- If $\widetilde{R}$ is a good approximation of R, the linear system produced after the discretization of (1) is expected to be well-conditioned

PLAN

Goal : The solution of electromagnetic scattering problems by an obstacle whose the surface is covered by thin layers of imperfectly conductor materials.



Example : paints

PLAN

Notations

- D is a bounded domain with a $\mathcal{C}^\infty$ boundary Γ
- Ω^+ is the exterior of D, k is the wave number in Ω^+
- The space of admissible waves is W^+

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W^+

- $\mathbf{E}$ is in Ω^+
- $\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = 0$
- $\mathbf{E}$ has a tangential trace on Γ
- $\mathbf{E}$ is a radiating field at infinity

Notations

- D is a bounded domain with a $\mathcal{C}^\infty$ boundary Γ
- Ω^+ is the exterior of D, k is the wave number in Ω^+
- The space of admissible waves is W^+
- Stratton-Chu formula valid for all $\mathbf{E} \in W^+$ is

$$\mathbf{E} = \mathcal{T}\mathbf{n} \times \mathbf{H} - \mathcal{K}\mathbf{n} \times \mathbf{E} \quad ,$$

where

- $\mathbf{H} = \dfrac{1}{ik} \nabla \times \mathbf{E}$
- $\mathcal{T} = \dfrac{1}{ik} \nabla \times \nabla \times \mathcal{G}$
- $\mathcal{K} = \nabla \times \mathcal{G}$
- $\mathcal{G}\mathbf{u}(x) = \dfrac{-1}{4\pi} \displaystyle\int_\Gamma \frac{e^{ik\|x-y\|}}{\|x-y\|} \mathbf{u}(y)\, dy$

Application to boundary value problems

The scattering problem with an impedance condition...

Given an incident field $\mathbf{E}^{inc}$, the problem is

$$\text{Find } \mathbf{E} \in W^{+} \text{ such that } \mathbf{E}_{\tan} + \alpha \mathbf{n} \times \mathbf{H} = -\mathbf{E}_{\tan}^{\mathrm{inc}} + \alpha \mathbf{n} \times \mathbf{H}^{\mathrm{inc}}$$

where α is a complex-valued function defined on Γ

...is embedded in the initial abstract problem

$$\text{Find } w \in W \text{ such that } \gamma w = u_0$$

Application to boundary value problems

The scattering problem with an impedance condition. . .

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. . . is embedded in the initial abstract problem

- The space W of admissible waves is W^{+}
- The boundary condition trace operator γ is $\gamma\,\mathbf{E} = \mathbf{n} \times \mathbf{E} - \alpha \mathbf{H}_{\tan}$
- The source excitation u_0 becomes $-\gamma \mathbf{E}^{\mathrm{inc}}$

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The R operator for Leontovich problems

- We have now to identify the (optimal) regularizing R operator
- R depends on a choice of a Cauchy data trace operator γ_c equipped with a Calderón potential $\mathcal{C}$ such that

$$\mathbf{E} = \mathcal{C}(\gamma_c \mathbf{E}) \qquad \forall\, \mathbf{E} \in W^+$$

- Because of the Statton-Chu formula, we choose

$$\gamma_c \mathbf{E} = (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \qquad\qquad \mathcal{C}(\mathbf{u}, \mathbf{v}) = \mathcal{L}\mathbf{v} - \mathcal{K}\mathbf{u}$$

- We want to give an expression of R more tractable than the definition

$$R: \ \gamma\mathbf{E} \mapsto \gamma_c \mathbf{E}$$

which requires to solve the initial boundary value problem.

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Towards an approximation of R

- From the choice made before $R : \mathbf{n} \times \mathbf{E} - \alpha\, \mathbf{H}_{\tan} \mapsto (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H})$
- Writing R in coordinates (R_E, R_H) one has

$$R_E \mathbf{u} = \mathbf{n} \times \mathbf{E} \qquad R_H \mathbf{u} = \mathbf{n} \times \mathbf{H} \tag{2}$$

$$\text{with } \mathbf{u} = \mathbf{n} \times \mathbf{E} - \alpha\, \mathbf{H}_{\tan} \tag{3}$$

- Expanding $\mathbf{n} \times \mathbf{E}$ and $\mathbf{H}_{\tan}$ in (3) with (2) gives $R_E = \mathrm{Id} - \alpha \mathbf{n} \times R_H$

$$R = (\mathrm{Id} - \alpha \mathbf{n} \times R_H, R_H)$$

- We suggest to approach R as

$$\widetilde{R} = (\mathrm{Id} - \alpha \mathbf{n} \times \widetilde{R}_H, \widetilde{R}_H)$$

where $\widetilde{R}_H$ is an approximation of R_H to build.

Towards an approximation of R_H

- Let Y_+ be the exterior admittance of Γ linking $\mathbf{n} \times \mathbf{E}$ to $\mathbf{n} \times \mathbf{H}$
- $\mathbf{u} = \mathbf{n} \times \mathbf{E} - \alpha\, \mathbf{H}_{\tan}$, $R_H \mathbf{u} = \mathbf{n} \times \mathbf{H}$ and $\mathbf{n} \times \mathbf{E} = -Y_+(\mathbf{n} \times \mathbf{H})$ give

$$R_H = (\alpha \mathbf{n} \times \mathrm{Id} - Y_+)^{-1}$$

- If $\alpha = 0$ (PEC problem) $R_H = Y_+$ (because $Y_+^2 = -\mathrm{Id}$)
- It seems natural to search an approximation of R_H under the form

$$\widetilde{R}_H = \begin{cases} (\alpha \mathbf{n} \times \mathrm{Id} - \widetilde{Y}_+)^{-1} \text{ if } \alpha \neq 0 \\ \widetilde{Y}_+ \text{ if } \alpha = 0 \end{cases}$$

Towards an approximation of R_H

- Let Y_+ be the exterior admittance of Γ linking $\mathbf{n} \times \mathbf{E}$ to $\mathbf{n} \times \mathbf{H}$
- $\mathbf{u} = \mathbf{n} \times \mathbf{E} - \alpha\, \mathbf{H}_{\tan}$, $R_H \mathbf{u} = \mathbf{n} \times \mathbf{H}$ and $\mathbf{n} \times \mathbf{E} = -Y_+(\mathbf{n} \times \mathbf{H})$ give

$$R_H = (\alpha \mathbf{n} \times \mathrm{Id} - Y_+)^{-1}$$

- If $\alpha = 0$ (PEC problem) $R_H = Y_+$ (because $Y_+^2 = -\mathrm{Id}$)
- Hence, the goal now is to find good approximations of Y_+ such that

$$\widetilde{R}_H = \begin{cases} (\alpha \mathbf{n} \times \mathrm{Id} - \widetilde{Y}_+)^{-1} \text{ if } \alpha \neq 0 \\ \widetilde{Y}_+ \text{ if } \alpha = 0 \end{cases}$$

leads to a well-posed GSIE equation

$$\mathbf{n} \times \mathrm{T}\widetilde{R}_H \mathbf{u} - (\mathbf{n} \times \mathrm{K} - \frac{1}{2}\mathrm{Id})(\mathbf{u} - \alpha \mathbf{n} \times \widetilde{R}_H \mathbf{u}) = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} + \alpha \mathbf{H}^{\mathrm{inc}}_{\tan}$$

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A direct approximation

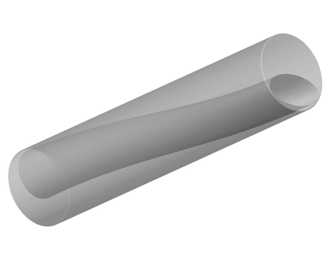
$$\text{Admittance of the tangent plane is } -2\mathbf{n}\times\mathcal{T} \tag{4}$$

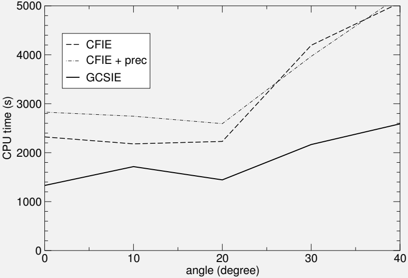
$$\Longrightarrow \widetilde{Y}_{+} = -2\sum_{p}\chi_p\,\mathbf{n}\times\mathcal{T}\,\chi_p \tag{5}$$

with $(U_p, \chi_p)_p$ a quadratic partition of the unity.

- When $\alpha = 0$ and under assumption on the width of patches (which have to be not too small compared to the wavelength), the GSIE is a compact perturbation of a positive operator
- The question to know if, when $\alpha \neq 0$, the GSIE is always well-posed with (5) is not answered for the moment
- The next construction of $\widetilde{Y}_{+}$ overcomes this problem.

Numerical results





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An indirect appr.*via* the Helmholtz potentials

There exit two boundary operators P_{loop}, P_{star} going from $H_{\mathrm{T}}(\Gamma)$ to $H(\Gamma)$ such that for all $\mathbf{u} \in \mathbf{H}_{\mathrm{T}}(\boldsymbol{\Gamma})$

$$\mathbf{u} = -\mathbf{n} \times \nabla \mathbf{P}_{\mathrm{loop}}\mathbf{u} + \nabla \mathbf{P}_{\mathrm{star}}\mathbf{u}$$

If A is an operator acting on vector fields of Γ, we can identify A with a 2×2 matrix of operators acting on scalar fields following

$$A = \begin{pmatrix} -\mathbf{n} \times \nabla & \nabla \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} P_{\mathrm{loop}} \\ P_{\mathrm{star}} \end{pmatrix} .$$

An indirect appr. *via* the Helmholtz potentials

- On a plane, the Helmholtz decomposition of $\mathbf{n} \times \mathcal{T}$ is

$$\mathbf{n} \times \mathcal{T} = \frac{1}{ik} \begin{pmatrix} 0 & -G(\Delta + k^2 \mathrm{Id}) \\ k^2 G & 0 \end{pmatrix} .$$

- Still in the plane, the Fourier transform of the kernel of G is $\hat{G}(\xi) = \frac{1}{2i}(k^2 - \|\xi\|^2)^{-1/2}$. Therefore $G = \frac{1}{2i}(\Delta + k^2 \mathrm{Id})^{-1/2}$
- And because $Y_+ = -2\mathbf{n} \times \mathcal{T}$ on the plane, Y_+ is equal to

$$Y_+ = \frac{1}{k} \begin{pmatrix} 0 & -(\Delta + k^2 \mathrm{Id})^{1/2} \\ k^2 (\Delta + k^2 \mathrm{Id})^{-1/2} & 0 \end{pmatrix} \qquad (6)$$

- if Δ in (6) is viewed as the Laplace-Beltrami operator, this formula is able to define a $\widetilde{Y}_+$ operator on Γ candidate to the GSIE.

An indirect appr. *via* the Helmholtz potentials

- Therefore, on a general surface Γ we suggest to take

$$\widetilde{Y}_{+} = \frac{1}{k}\begin{pmatrix} 0 & -(\Delta + k^2\mathrm{Id})^{1/2} \\ k^2(\Delta + k^2\mathrm{Id})^{-1/2} & 0 \end{pmatrix}$$

- But as in the direct technique, the GSIE suffers from spurious modes
- Equivalent in spirit to the localization process used with the cut-off functions, we have to localize $\widetilde{Y}_{+}$ in replacing k with $k + i\epsilon$ where ϵ is a small damping parameter
- Hence, we obtain a well-posed equation being furthermore a compact perturbation of identity on $L^2_{\mathrm{T}}(\Gamma)$

Discretization of the GSIE

The GSIE equation (indirect method)

$$\mathbf{n} \times T\widetilde{R}_H\mathbf{u} - (\mathbf{n} \times K - \tfrac{1}{2}\mathrm{Id})(\mathbf{u} - \alpha\mathbf{n} \times \widetilde{R}_H\mathbf{u}) = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} + \alpha\mathbf{H}^{\mathrm{inc}}_{\mathrm{tan}}$$

$$\widetilde{R}_H = (\alpha\mathbf{n} \times \mathrm{Id} - \widetilde{Y}_+)^{-1}$$

$$\widetilde{Y}_+ = \frac{1}{k}\begin{pmatrix} 0 & -(\Delta + k^2\mathrm{Id})^{1/2} \\ k^2(\Delta + k^2\mathrm{Id})^{-1/2} & 0 \end{pmatrix}$$

- The problem to overcome is the synthesis of the square roots in $\widetilde{Y}_+$.
- The technique used is based on a Padé expansion of the square root
- In practice, it leads to the construction of some additional sparse matrices we have to factorize
- The evaluation of $\widetilde{R}_H$ represents at most 20% of the total CPU time.

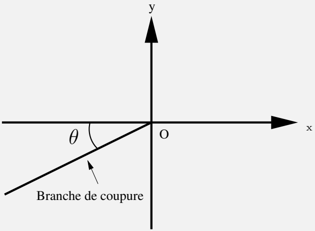
Influence of the order p of the Padé approximation

$$\left(1+\frac{\Delta_\Gamma}{k_\varepsilon^2}\right)^{\frac{1}{2}}$$ is a non-local pseudo-differential operator.

Approximation : We use a Padé approximation based on a rotating branch-cut technique (θ is the angle of the rotation)

$$\left(1+\frac{\Delta_\Gamma}{k_\varepsilon^2}\right)^{\frac{1}{2}} =$$

$$A_0+\sum_{j=1}^{p} A_j \frac{\Delta_\Gamma}{k_\varepsilon^2}\left(I+B_j\frac{\Delta_\Gamma}{k_\varepsilon^2}\right)^{-1}$$



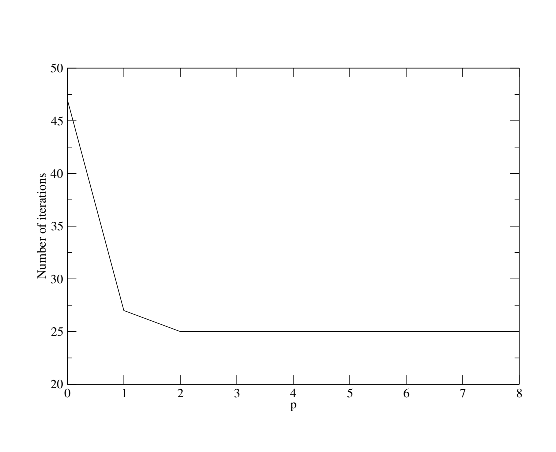
Influence of the order p of the Padé approximation

Goal : Influence of p on the convergence of the iterative solver and the accuracy of the solution

Test-case :

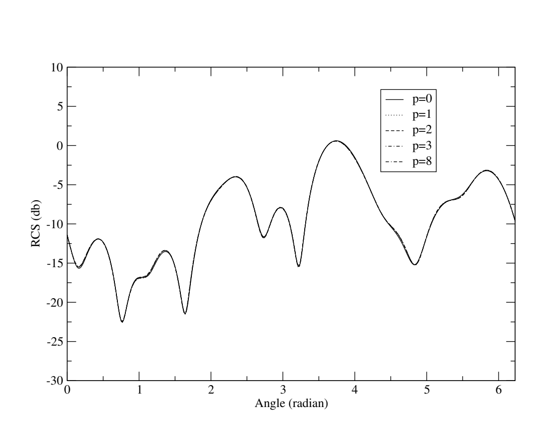
- Cube partially covered ($\eta = 1.$ on a face) whose the mesh is composed of 8460 edges and $k = 20$.
- The incident plane wave goes on a corner of the cube where the impedance is discontinuous.

Influence of the order p of the Padé approximation



Number of iterations in function of p for a residual equals to 10^{-8}

Influence of the order p of the Padé approximation



RCS in function of p for a residual equals to 10^{-8}

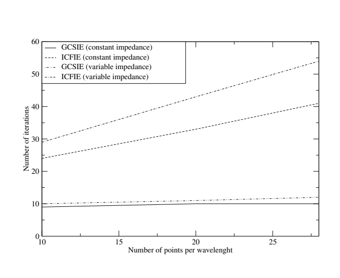
Numerical results : sphere

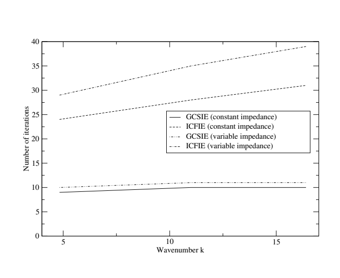
Test-case : A sphere of radius $1m$.
The meshes used are composed of 1500, 6000 and 13500 edges which respectively correspond to wavenumbers k equal to 4.83, 11 and 16.4. (≈ 10 points per wavelenght).

Two situations :

- Constante impedance : $\eta = 0.34$,
- Variable impedance : $\eta = 0.34$ on $\Gamma_i = \{(x, y, z) \; : \; x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$.

Numerical results : sphere





Behavior for the space refinement (left) and the frequency increase (right)

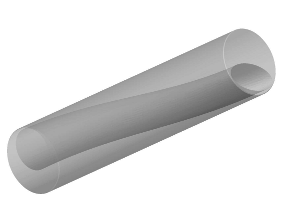
Comparison with other equations

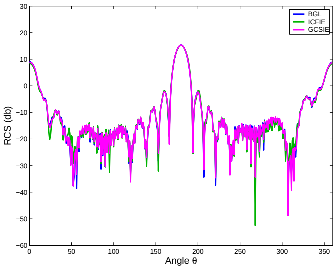
- The Impedant CFIE (ICFIE)

  F. Collino, F. Millot, and S. Pernet. *Boundary-integral methods for iterative solution of scattering problems with variable impedance surface condition*, PIER **80** (2008), 1–28.

- The Bachelot-Gay-Lange integral equation (BGLIE)

  V. Lange. *Equations intégrales espace-temps pour les équations de Maxwell. Calcul du champ diffracté par un obstacle dissipatif*. PhD thesis, Université de Bordeaux, 1995.



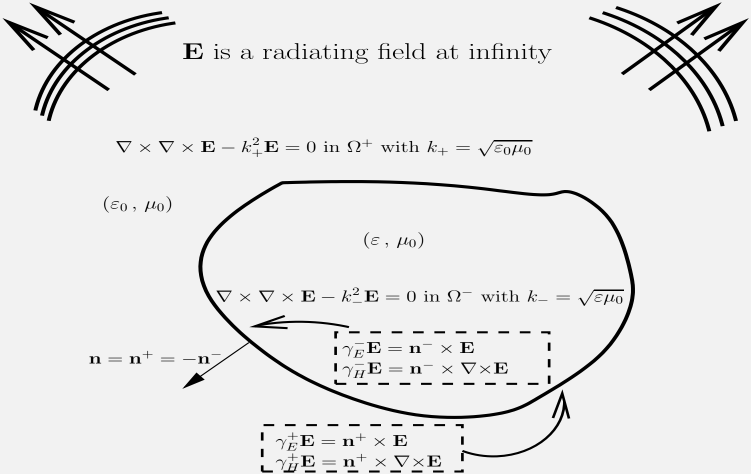


Numerical results : iteration counts & CPU times

- Channel, full or partially coated at 5GHz with 153,033 unknowns
- Computation on a cluster of 8 processors AMD Opteron 2.4 GHz (2 Go RAM)
- Iterative solvers is GMRES (or Flexible GMRES), possibly combined with a SPAI preconditioner
- Stopping criterion on residue is fixed to 10^{-4}
- We succeeded in dividing by 3 the numerical cost of computation

Equation	Solver	Coating	Iterations	CPU time
GSIE	GMRES	All surface	22	$43min$
BGLIE	GMRES + prec	All surface	No convergence	$14h12min$
BGLIE	FGMRES	All surface	18	$21h32min$
ICFIE	GMRES + prec	All surface	35	$3h$
GSIE	GMRES	Inner surface	22	$43min$
BGLIE	FGMRES	Inner surface	44	$37h$
ICFIE	GMRES + prec	Inner surface	37	$3h$

Model problem and aditional notations



Space of admissible waves

The space of admissible waves is W of all electric fields $\mathbf{E}$ defined on $\mathbb{R}^3 \backslash \Gamma$

$W = W^+ \oplus W^-$

- $\mathbf{E}$ is in Ω^+
- $\nabla \times \nabla \times \mathbf{E} - k_+^2 \mathbf{E} = 0$
- $\mathbf{E}$ has a tangential trace on Γ
- $\mathbf{E}$ is a radiating field at infinity

- $\mathbf{E}$ is in Ω^-
- $\nabla \times \nabla \times \mathbf{E} - k_-^2 \mathbf{E} = 0$
- $\mathbf{E}$ has a tangential trace on Γ

The transmission problem

The transmission problem

Find $\mathbf{E} \in W$ such that $\begin{cases} \mathbf{n}^+ \times \mathbf{E} + \mathbf{n}^- \times \mathbf{E} & = -\mathbf{n}^+ \times \mathbf{E}^{\mathrm{inc}} \\ \mathbf{n}^+ \times \nabla \times \mathbf{E} + \mathbf{n}^- \times \nabla \times \mathbf{E} & = -\mathbf{n}^+ \times \nabla \times \mathbf{E}^{\mathrm{inc}} \end{cases}$ (7)

$\cdots$ is embedded in the intial abstract problem

- $\gamma = (\gamma^+_{EH'} + \gamma^-_{EH'})$ with $\gamma^+_{EH'} = (\gamma^+_E, \gamma^+_{H'})$, $\gamma^-_{EH'} = (\gamma^-_E, \gamma^-_{H'})$
- $\gamma\mathbf{E} = (\mathbf{u}_0, \mathbf{v}_0) = -\gamma^+_{\mathbf{EH'}}\mathbf{E}^{\mathrm{inc}}$.
- Cauchy data trace operator is choosen as $\gamma_c = (\gamma^+_{EH'}, \gamma^-_{EH'})$
- the Calderón potential $\mathcal{C}(\mathbf{u}^+, \mathbf{v}^+, \mathbf{u}^-, \mathbf{v}^-) = \mathcal{V}^+(\mathbf{u}^+, \mathbf{v}^+) + \mathcal{V}^-(\mathbf{u}^-, \mathbf{v}^-)$, with $\mathcal{V}_\pm(\mathbf{u}, \mathbf{v}) = \dfrac{1}{ik^\pm}\mathcal{T}_\pm\mathbf{v} - \mathcal{K}_\pm\mathbf{u}$.

The R operator and GSIE for the transmission problems

- The transmission problem is well-posed $\Rightarrow \exists \mathcal{R}$ such that $\mathcal{R}(\mathbf{u_0}, \mathbf{v_0}) = \mathbf{E}$. In particular, $(\gamma^+_{EH'} + \gamma^-_{EH'})\mathcal{R} = \mathrm{Id}$ and therefore if $R^+ = \gamma^+_{EH'}\mathcal{R}$ and $R^- = \gamma^-_{EH'} R$ then

$$R^+ + R^- = \mathrm{Id}. \tag{8}$$

Remark : R^+ and R^+ verify : $R^\pm \circ R^\pm = R^\pm$ and $R^\pm \circ R^\mp = 0$ and split $\mathcal{D}'(\Gamma) \times \mathcal{D}'(\Gamma)$ in a direct sum.

- If we define $R = (R^+, \mathrm{Id} - R^+)$, then we have :

$$\mathbf{E} = \mathcal{C} \circ R(\mathbf{u_0}, \mathbf{v_0}) = \mathcal{C}(\mathbf{u}_0^+, \mathbf{v}_0^+, \mathbf{u}_0^-, \mathbf{v}_0^-) \quad \Rightarrow \quad \gamma\mathcal{C} \circ R = \mathrm{Id}$$

- Noticing $C^+ = \gamma^+_{EH'}\mathcal{C}$ and $C^- = \gamma^-_{EH'}\mathcal{C}$, we obtain the crucial relation :

$$\left(C^+ R^+ + C^-(\mathrm{Id} - R^+)\right) = \mathrm{Id} \quad . \tag{9}$$

The R operator and GSIE for the transmission problems

- If $\widetilde{R}_+$ is an approximation of R^+, it is natural to take $\mathrm{Id} - \widetilde{R}_+$ as an approximation of $\widetilde{R}_-$.
- We take $\widetilde{R} = (\widetilde{R}^+, \mathrm{Id} - \widetilde{R}^+)$ and we parametrize the space W in this way :

$$\mathbf{E} \quad = \quad \mathcal{C} \circ \widetilde{R}(\mathbf{u}, \mathbf{v})$$

- So, we obtain the GSIE equation

$$\left(C^+ \widetilde{R}^+ + C^- (\mathrm{Id} - \widetilde{R}^+) \right) (\mathbf{u}, \mathbf{v}) = -\gamma^+_{EH'} \mathbf{E}^{\mathrm{inc}} \quad . \tag{10}$$

- Related to the Cauchy data $\gamma^+_{EH'}$ and $\gamma^-_{EH'}$ are the admittance operators

$$Y'_\pm : \mathbf{n}^\pm \times \mathbf{E} \mapsto \mathbf{n}^\pm \times \nabla\times\mathbf{E}$$

- Reading $R^\pm$ as a 2×2 matrix of operators : $R^\pm = \begin{pmatrix} R^\pm_{11} & R^\pm_{12} \\ R^\pm_{21} & R^\pm_{22} \end{pmatrix}$, it

  leads to the relations :
  - $R^+_{11} + R^-_{11} = \mathrm{Id}$, $R^+_{22} + R^-_{22} = \mathrm{Id}$, $R^+_{12} = -R^-_{12}$ and $R^+_{21} = -R^-_{21}$

- $\begin{pmatrix} R^\pm_{11} & R^\pm_{12} \\ R^\pm_{21} & R^\pm_{22} \end{pmatrix} \begin{pmatrix} \mathbf{u_0} \\ \mathbf{v_0} \end{pmatrix} = \begin{pmatrix} \mathbf{n}^\pm \times \mathbf{E} \\ Y'_\pm(\mathbf{n}^\pm \times \mathbf{E}) \end{pmatrix} \Rightarrow \begin{cases} Y'_\pm R^\pm_{11} = R^\pm_{21} \\ Y'_\pm R^\pm_{12} = R^\pm_{22} \end{cases}$

which give us

$$R^+ = \begin{pmatrix} A & -AZ'_- \\ Y'_+A & -Y'_+AZ'_- \end{pmatrix} . \qquad (11)$$

where

$$A = -(Y'_+ - Y'_-)^{-1} Y'_- \quad \text{and } Z'_- = {Y'_-}^{-1} \qquad (12)$$

An approximation $\widetilde{R}_+$

- $$\widetilde{R}^+ = \begin{pmatrix} \widetilde{R}_E^+ \\ \widetilde{Y}'_+ \widetilde{R}_E^+ \end{pmatrix} \text{ where } \widetilde{R}_E^+ = \widetilde{A} \begin{pmatrix} \mathrm{Id} & -\widetilde{Z}'_- \end{pmatrix} .$$

- As approximation of A we take

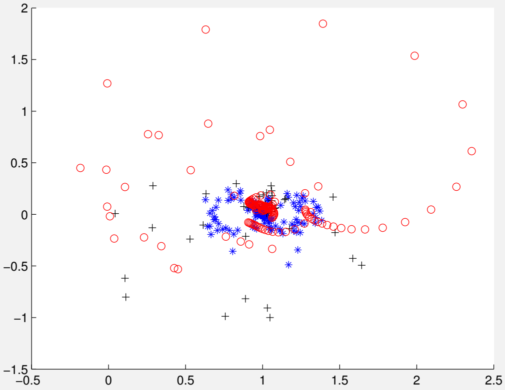
$$\widetilde{A} = \frac{1}{\alpha^2+1}\left(\mathrm{Id} + \frac{\alpha^2-1}{2}\Pi_{\mathrm{star}} \right)$$

with $\alpha^2 = k^+/k^-$ and $\Pi_{\mathrm{star}} = \nabla\Delta^{-1}\nabla\cdot$.

- $Y'_+ = ik^+ Y_+$ and $Z'_- = (ik^- Y_-)^{-1} = -\frac{1}{ik^-} Y_-$ therefore, Y'_+ and Z'_- can be approached with the previous technique.
- the resulting GSIE is a well-posed equation at any frequency : the underlying operator is a one-to-one mapping and a compact perturbation of identity in $H^{-1/2}_{\mathrm{div}} \cap L^2_{\mathrm{T}}$.

Spectrum on a sphere

Transmission problem : analytical results on a sphere for $k^{+} = 50$ and $k^{-} = 70.71$. Spectrum of the GCSIE.



GMRES convergence on a sphere

Transmission problem : analytical results on a sphere for $k^{+} = 50$ and $k^{-} = 70.71$. GMRES convergence historical of several formulations.